

5. Groups and Algebras

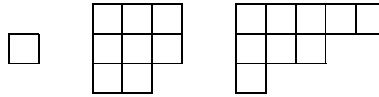
1. Some Algebraic Algebra

Show that

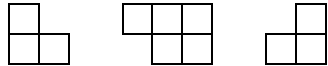
- (i) $f_{abc}T^bT^c = \frac{i}{2}C_A T^a$
- (ii) $T^bT^aT^b = (C_F - \frac{1}{2}C_A)T^a$.

2. Young Tableaux

Young tableaux are graphical representations of *irreps* that correspond to tensors. Tableaux (the singular is tableau) are drawn as connected boxes. For example, the fundamental rep is drawn as \square . A rank n symmetric tensor is written as a row of n boxes: $S^{ijk} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$ whereas a rank n antisymmetric tensor is a column of n boxes, eg, $A^{ij} = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$. A general tableau will be of mixed symmetry. A tableau can consist of any number of boxes provided that no more than N boxes occur in any column (for $SU(N)$), the rows start at the left, and no row is longer than a row above it. Thus the following are valid tableaux



whereas the following are not



The dimension of an $SU(N)$ irrep is given by the formula

$$D = \prod_{\text{boxes}} \frac{(N + d_{\text{box}})}{h_{\text{box}}}$$

where d_{box} is assigned as shown here:

0	1	2	3
-1	0	1	2
-2	-1	0	1

etc. The h_{box} are called the *hook lengths* and are given by one plus the number of boxes to the right of the box plus the number of boxes below the box. For example, hook lengths for $\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$ are given by

4	2	1
1		

Thus, in $SU(N)$ we have the following dimensions (for example):

column of $N - 1$ boxes with an additional box at the top of the second column, and the singlet, which is a column of N boxes.

(a) Compute $F \otimes F$, $F \otimes A$, and $A \otimes A$ for $SU(2)$. Confirm that your results agree with the spin algebra familiar from undergrad.

(b) Compute $3 \otimes \bar{3}$, $3 \otimes 3$, $3 \otimes 3 \otimes 3$, $3 \otimes 6$, and $6 \otimes 6$ in $SU(3)$.

(c) Compute $F \otimes F$, $\bar{20} \otimes \bar{4}$ in $SU(4)$. Use $\bar{20} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$ and $\bar{4} = \begin{array}{|c|} \hline \square \\ \hline \end{array}$.

(d) Show that $F \otimes F$ is always the sum of a symmetric and an antisymmetric rep. Show that $F \otimes \bar{F} = 1 + A$.

3. Adjoint Fermions

Write down the Lagrangian density for $SU(N)$ gauge fields coupled to fermions in the adjoint representation. Don't forget the gauge fixing and ghost terms.