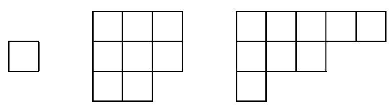
4. Groups and Algebras

1. Some Algebraic Algebra

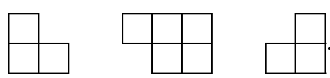
Show that
(i) $f_{abc}T^bT^c = \frac{i}{2}C_AT^a$
(ii) $T^bT^aT^b = \left(C_F - \frac{1}{2}C_A\right)T^a$.

2. Young Tableaux

Young tableaux are graphical representations of *irreps* that correspond to tensors. Tableaux (the singular is tableau) are drawn as connected boxes. For example, the fundamental rep is drawn as $\square$. A rank n symmetric tensor is written as a row of n boxes: $S^{ijk} = \boxed{i}\boxed{j}\boxed{k}$ whereas a rank n antisymmetric tensor is a column of n boxes, eg, $A^{ij} = \begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$. A general tableau will be of mixed symmetry. A tableau can consist of any number of boxes provided that no more than N boxes occur in any column (for SU(N)), the rows start at the left, and no row is longer than a row above it. Thus the following are valid tableaux



whereas the following are not

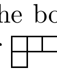


The dimension of an SU(N) irrep is given by the formula

$$D = \prod_{\text{boxes}} \frac{(N + d_{\text{box}})}{h_{\text{box}}}$$

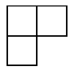
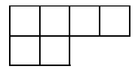
where d_{box} is assigned as shown here:

0	1	2	3
-1	0	1	2
-2	-1	0	1

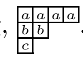
etc. The h_{box} are called the *hook lengths* and are given by one plus the number of boxes to the right of the box plus the number of boxes below the box. For example, hook lengths for  are given by

4	2	1
1		

Thus, in SU(N) we have the following dimensions (for example):

 $= N$  $= \frac{1}{3}N(N+1)(N-1)$  $= \frac{1}{80}N^2(N^2-1)(N+2)(N+3)$

Finally, tableaux can be combined to determine irreps of products of irreps, say $T_1 \otimes T_2$. To do this

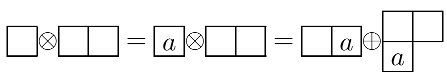
(i) assign distinct labels to boxes in each row of T_1 as follows, eg, .

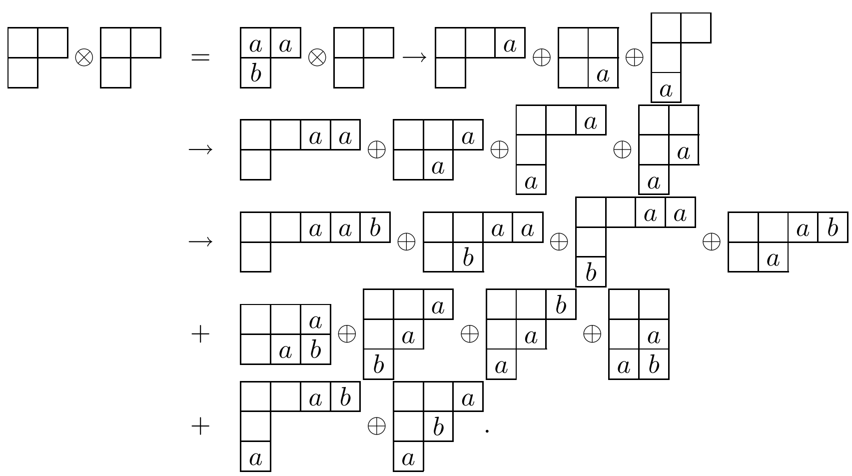
(ii) attach boxes labelled by a to T_2 in all possible ways such that no two as appear in the same column and the result is still a Young tableau.

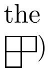
(iii) repeat with bs, cs, etc

(iv) after all the boxes of T_1 have been added to T_2 form the *sequence* for each tableau by reading each row from right to left while going from top to bottom. Eliminate tableaux with inadmissable sequences, namely, when reading the sequence (from left to right) there must be at least as many as as bs, bs as cs, etc at any point in the sequence. Any resulting tableau with the same structure and sequence are copies of each other and only one tableau should be considered.

Some examples:

$N \otimes \frac{1}{2}N(N+1) =$  $= \frac{1}{6}N(N+1)(N+2) + \frac{1}{3}N(N^2-1).$



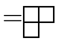
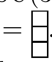
The sequences for these are *baa*, *aab*, *aab*, *baa*, *aba*, *aab*, *baa*, *aba*, *baa*, *aba*. All are admissable except *baa* so we remove tableaux 1, 4, 7, and 9. The remaining have the following dimensions in SU(3): 27, 10, 10 (a different 10!), 8 (different from ), 1, 8 (notice another way to write the singlet). Thus

$$8 \otimes 8 = 1 \oplus 8' \oplus 8' \oplus 10 \oplus 10' \oplus 27.$$

Example irreps are the fundamental, $F =$ , the conjugate of the fundamental, $\bar{F}$, which is represented as a column of $N-1$ boxes, the adjoint, which is a column of $N-1$ boxes with an additional box at the top of the second column, and the singlet, which is a column of N boxes.

(a) Compute $F \otimes F$, $F \otimes A$, and $A \otimes A$ for SU(2). Confirm that your results agree with the spin algebra familiar from undergrad.

(b) Compute $3 \otimes \bar{3}$, $3 \otimes 3$, $3 \otimes 3 \otimes 3$, $3 \otimes 6$, and $6 \otimes 6$ in SU(3).

(c) Compute $F \otimes F$, $\bar{20} \otimes \bar{4}$ in SU(4). Use $\bar{20} =$  and $\bar{4} =$ .

(d) Show that $F \otimes F$ is always the sum of a symmetric and an antisymmetric rep. Show that $F \otimes \bar{F} = 1 + A$.

3. Adjoint Fermions

Write down the Lagrangian density for SU(N) gauge fields coupled to fermions in the adjoint representation. Don't forget the gauge fixing and ghost terms.