2. Functional Approach to Field Theory

1. Schwinger-Dyson Equation

We derived

$$\int d^4x \left[\delta^4(0)F'(\frac{\delta}{i\delta J(x)}) + i\left(\lim_{y\to x}\frac{\delta S}{\delta\phi}\left(\frac{\delta}{i\delta J(y)}\right) + J(x)\right)F(\frac{\delta}{i\delta J(x)})\right]Z[J] = 0$$

for an interacting scalar field theory. Set $V = 0$ and $F(\chi) = \frac{1}{3}\chi^3$ and show that the equation is satisfied. How do things go wrong if the limit is not used?

2. Discretised Functional Derivatives

Obtain a discrete form of the functional derivative $\frac{\delta}{\delta f(x)}$.

3. The Schrödinger Functional Formalism

Derive H for a free scalar field using the path integral method we employed for the quantum mechanical case. Interpret your result.

4. Scalar Field Vacuum Wavefunctional

Using the Ansatz (ω is to be determined)

$$\langle\phi|\Psi_0\rangle \propto \mathrm{e}^{-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\phi(k)\omega(k)\phi(-k)}$$

and your Hamiltonian from question 3 obtain the exact ground state wavefunctional for free (neutral) scalar field theory. Derive the ground state energy density. You will need the formula

$$E_0 = \frac{\langle\Psi_0|H|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}$$

and

$$\langle\Psi|\mathcal{O}|\Lambda\rangle = \int D\phi\Psi^*[\phi]\mathcal{O}\Lambda[\phi].$$

5. Variational Wavefunctional

Obtain an estimate of the ground state energy density for ϕ^4 theory using the Ansatz of question 4.