

1. Subtleties with Path Integrals

1. Free Particle

Set

$$U(x_0 x_f; t + \epsilon) = \int \frac{dx}{C} e^{\frac{im}{\hbar\epsilon}(x_f - x)^2} U(x_0, x; t),$$

$\eta = x_f - x$, expand to order $\epsilon = \eta^2$, and obtain the expected Schrödinger equation.

2. A Corollary

When integrating over angular variables one has expressions such as

$$\int_{-\pi/2}^{\pi/2} \frac{d\theta}{C} e^{ia\theta^2/\epsilon} \frac{\theta^{2m}}{\epsilon^{m-1}},$$

which is $O(\epsilon)$. But our integrals have always been from $-\infty$ to ∞ . Show that extending the region of integration only changes the result at higher order in ϵ by making the change of variables $x = \tan \theta$ and expanding. You will need this in the following.

3. Discretisation Issues: Curvilinear Coordinates

(i). Problems exist when attempting even simple generalisations. A simple example is as follows: let $y(t) = x(t)^3$ so that $\dot{y} = 3x^2\dot{x}$. Discretising this expression as $(y_{i+1} - y_i)/\epsilon$ gives

$$\dot{y} = (x_{i+1}^3 - x_i^3)/\epsilon.$$

But one could equally well write (discretising $3x^2\dot{x}$):

$$\dot{y} = 3 \left(\frac{x_{i+1} + x_i}{2} \right)^2 \left(\frac{x_{i+1} - x_i}{\epsilon} \right).$$

These expressions do not agree. Find a discretisation of x^2 that makes them agree.

(ii). Now consider a free particle in two dimensions in angular coordinates:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2).$$

Compute H by considering the time evolution from r_0, θ_0, t to $R, \Theta, t + \epsilon$ using the discretisation of $L(r, \theta)$ above.

(iii). Compute H again, but this time use the discretisation of $L(x, y)$, written in angular variables. Comment on your results.

4. Operator Ordering Ambiguities

A simple classical problem with the operator ordering ambiguity is given by $L = \frac{1}{2}m(x)\dot{x}^2$. In this case the path integral normalisation depends on x : $C \rightarrow C(x)$ and one must choose how to discretise this. Furthermore, there is no obvious discretisation of the action because we do not know if potential terms are present in the quantum version of L (due to the operator ordering ambiguity). We model this situation by setting

$$U(x_0, x_f; t + \epsilon) = \int \frac{dx}{C(ux_f + (1-u)x)} e^{i/\hbar S(x \rightarrow x_f)} U(x_0, x; t)$$

and discretise S as

$$S(x \rightarrow x_f) = \epsilon \frac{1}{2} m(vx_f + (1-v)x) \frac{\eta^2}{\epsilon^2}$$

with $\eta = x_f - x$.

- (i). Determine H .
- (ii). What are u and v if H is to be Hermitian?
- (iii). Weyl ordering is defined to be

$$WO(f(x)g(p)) = e^{-\frac{1}{2}i\hbar \frac{\partial^2}{\partial x \partial p}} f(x)g(p)$$

Determine $WO(\frac{1}{2m(x)}p^2)$. Finally, obtain the values of u and v that correspond to Weyl ordering.

NB: Many other discretisation procedures are possible! The path integral does not resolve the operator ordering ambiguity.

5. Constrained Path Integrals

Consider the motion of a free particle constrained to move on a circle of radius a . The path integral can be written as

$$U(\theta_f, \theta_i; t) = \int D\vec{x} e^{\frac{i}{\hbar} S}$$

with

$$D\vec{x} \propto \prod dx_i dy_i \delta(x_i^2 + y_i^2 - a^2) \propto \prod dx_i dy_i d\lambda_i e^{i\epsilon \lambda_i (x_i^2 + y_i^2 - a^2)}.$$

Derive the expression for H using this formalism. Draw a typical path on the constrained surface in xyt space. Do you see a reason for the peculiar result for H ?

Bonus: a closed space such as this permits ‘topological paths’. See if you can figure out what this means in this context. How do your results change? (no need to compute, just discuss).

NB: In general one must ensure that the correct measure on the constrained surface is obtained, so it is not sufficient to simply insert a delta function. Specifically, if $F(x_1, x_2, \dots, x_D)$ specifies the constraint surface then one wants

$$\int dx_1 dx_2 \dots dx_D \delta(F) \Delta(x_1, \dots, x_{D-1}) = \int dx_1 dx_2 \dots dx_{D-1} \sqrt{\det(g_{\mathcal{M}})}$$

where $g_{\mathcal{M}}$ is the metric induced on the constrained surface. This is just the Faddeev-Popov trick. Δ is a constant in the case above, so your results do not need to be modified.