Schwinger-Dyson Equations

Peskin and Schroeder, chapter 11.
Refs. Itzykson and Zuber, *Quantum Field Theory*, sections 6-2-2, 9-2-1, 10-1.
D.J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena*, chapter 5.

Schwinger-Dyson equations represent nonperturbative information on interacting quantum field theory. They are derived by taking derivatives with respect to the generating functional, and are basically equivalent to integration by parts.

One Particle Irreducible Diagrams

SDEs are usually derived in terms of 'one-particle-irreducible' (1PI) diagrams.
definition: 1PI diagrams (called 'proper' by I& Z) are truncated, connected diagrams which remain connected when any internal line is removed.

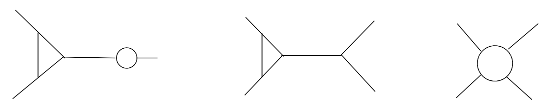


FIG. 1: (left) not truncated, not 1PI, (centre) truncated, not 1PI, (right) truncated, 1PI

1PI diagrams can be used to generate general diagrams and it can be shown that renormalising 1PI diagrams renormalises the entire theory. For example, if an open circle represents a 1PI vertex then the full two-point function can be written as shown here.

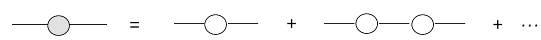


FIG. 2: Full two-point function in terms of the two-point 1PI diagram.

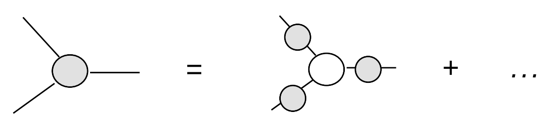


FIG. 3: Full three-point function in terms of the three-point 1PI diagram.

Legendre Transformation

The generator of connected diagrams is denoted F where

$$e^{iF[J]} = Z[J] = \int D\phi e^{iA+i\int J\phi}$$

(notice that I have an extra factor of i in my definition with respect to I & Z).
It is possible to generate 1PI diagrams via the Legendre transformation. This is done with the aid of a classical field defined by

$$\phi_c(x, J) = \frac{\delta}{\delta J(x)} F[J] = \langle\phi\rangle_J. \tag{1}$$

This is expectation value of the quantum field in the presence of the source J. We now define the generator of 1PI diagrams as (Jona-Lasinio, N. Cim. **34A**, 1790 (1964))

$$\Gamma[\phi_c] = F[J] - \int d^4x J(x)\phi_c(x).$$

Taking the derivative with respect the classical field and using the chain rule gives

$$\frac{\delta\Gamma}{\delta\phi_c(x)} = \int d^4y \frac{\delta F}{\delta J(y)}\frac{\delta J(y)}{\delta\phi_c(x)} - J(x) - \int d^4y \phi_c(y)\frac{\delta J(y)}{\delta\phi_c(x)}.$$

Using Eq. 1 then gives the central result

$$J(x) = -\frac{\delta\Gamma}{\delta\phi_c(x)} \tag{2}$$

Equations 1,2 allow one to relate derivatives of Γ to derivatives of F and to physical correlation functions. For example, the two-point function (propagator) is

$$S(x-y) = \langle\phi(x)\phi(y)\rangle = \frac{\delta^2}{\delta J(x)\delta J(y)}\log Z[J]|_{J=0} = i\frac{\delta^2}{\delta J(x)\delta J(y)}F|_{J=0}. \tag{3}$$

Notice that setting $J = 0$ implies that $\langle\phi\rangle_J = 0$ and thus one should set $\phi_c = 0$ when evaluating on-shell 1PI relationships.

We now consider

$$\delta(x-z) = \frac{\delta\phi_c(x)}{\delta\phi_c(z)} = \int d^4y \frac{\delta^2 F}{\delta J(x)\delta J(y)}\frac{J(y)}{\delta\phi_c(z)} = -\int d^4y \frac{\delta^2 F}{\delta J(x)\delta J(y)}\frac{\delta^2\Gamma}{\delta\phi_c(z)\delta\phi_c(y)} \tag{4}$$

Thus

$$\frac{\delta^2\Gamma}{\delta\phi_c(x)\delta\phi_c(z)} = -\left(\frac{\delta^2 F}{\delta J(x)\delta J(z)}\right)^{-1} \tag{5}$$

and

$$\frac{\delta^2\Gamma}{\delta\phi_c(x)\delta\phi_c(z)}|_{\phi_c=0} = iS^{-1}(x-z) \tag{6}$$

Similarly, the 1PI vertex is given by

$$\Gamma^{(3)}(x,y,z) = \frac{\delta^3\Gamma}{\delta\phi_c(x)\delta\phi_c(y)\delta\phi_c(z)}|_{\phi_c=0}. \tag{7}$$

Schwinger-Dyson Equations

As a first example we consider the Schwinger-Dyson equations for ϕ^3 theory. These may be derived by starting with the expression

$$\int D\phi \left[\frac{\delta S}{\delta\phi} + J\right] \mathrm{e}^{iS+i\int J\phi} = 0 \tag{8}$$

Use

$$\frac{\delta S}{\delta\phi(x)} = -(\partial_x^2 + m^2)\phi(x) - \frac{\lambda}{2}\phi^2(x)$$

Thus

$$\left[J(x) - (\partial_x^2 + m^2)\frac{\delta}{i\delta J(x)} - \frac{\lambda}{2}\left(\frac{\delta}{i\delta J(x)}\right)^2\right] Z[J] = 0$$

Now set $Z = \exp(iF)$ and premultiply by $\exp(-iF)$:

$$J(x) - (\partial_x^2 + m^2)\frac{\delta F}{\delta J(x)} + i\frac{\lambda}{2}\frac{\delta^2 F}{\delta J(x)^2} - \frac{\lambda}{2}\left(\frac{\delta F}{\delta J(x)}\right)^2 = 0$$

We can use the expressions (1), (2), and (5) to obtain:

$$-\frac{\delta\Gamma}{\delta\phi_c(x)} - (\partial^2 + m^2)\phi_c(x) - \frac{\lambda}{2}\phi_c(x)^2 - i\frac{\lambda}{2}\int d^4z \left(\frac{\delta^2\Gamma}{\delta\phi_c(x)\delta\phi_c(z)}\right)^{-1}\delta(x-z) = 0. \tag{9}$$

This equation represents the starting point for deriving all Schwinger-Dyson equations for ϕ^3 theory. For example, taking one more derivative gives

$$-\frac{\delta^2\Gamma}{\delta\phi_c(x)\delta\phi_c(y)} = (\partial_x^2+m^2)\delta(x-y)+\lambda\phi_c(x)\delta(x-y)-i\frac{\lambda}{2}\int d^4z\, d^4a\, d^4b \left(\frac{\delta^2\Gamma}{\delta\phi_c(x)\delta\phi_c(a)}\right)^{-1} \frac{\delta^3\Gamma}{\delta\phi_c(a)\delta\phi_c(y)\delta\phi_c(b)} \left(\frac{\delta^2\Gamma}{\delta\phi_c(b)\delta\phi_c(z)}\right)^{-1} \tag{10}$$

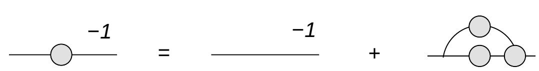
Here we have used the matrix equation

$$\frac{d}{dx}M^{-1} = -M^{-1}(\frac{dM}{dx})M^{-1}$$

which you can easily derive. Now evaluate at $\phi_c = 0$ to get the Schwinger-Dyson equation for the propagator:

$$-iS^{-1}(x-y) = (\partial_x^2 + m^2)\delta(x-y) + i\frac{\lambda}{2}\int d^4a\, d^4b\, S(z-a)\Gamma^{(3)}(a,y,b)S(b-x) \tag{11}$$

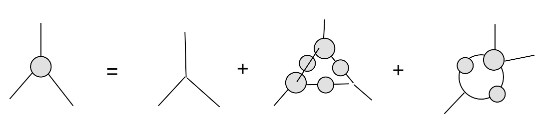
Diagrammatically, this can be written as



Note that there are no external legs in last term and that the full vertex is involved. Cross-multiplying rearranges this equation to read

where a full propagator and a bare propagator are now attached to the external legs of the last term.

This is not a closed integral equation for the full propagator because the full vertex is not known. It can be derived from Eq. 10 by taking another derivative and setting $\phi_c = 0$. The result in diagrammatic form is



Notice the appearance of a full four-point function in the last term. This pattern persists: the equation for the N-point function involves the N+1-point function. The Schwinger-Dyson equations are an infinite collection of integral equations that are equivalent to the full field theory.

The Effective Action

The classical vacuum is a constant for translationally invariant states. Also, Γ is an extensive quantity so one can write

$$\Gamma[\phi_c] = -(TL^3)V_{eff}(\phi_c)$$

where the prefactor is the spacetime volume and we have defined the 'effective potential'. Recall $\delta\Gamma/\delta\phi_c = -J$ so that

$$\frac{\partial V_{eff}}{\partial \phi_c}|_{J=0} = 0.$$

Finally, V_{eff} represents the energy density of the state in question.

[See PS section 11.3 for a discussion of the Maxwell construction in the effective potential. The upshot is that an unstable equilibrium point will be wiped out by states that are *not* spatially invariant.]

At tree order the effective potential is given by $V_{eff} = V_{int}(\phi_c)$. Quantum corrections can be computed in a clear way with the functional formalism. For example, we consider the computation of $Z[J]$ for ϕ^4 theory to one-loop order (for more details see PS section 11.4, where symmetry breaking and renormalisation are also considered). We let

$$\phi = \phi_c + \eta$$

where the classical vacuum satisfies

$$\frac{\delta S}{\delta \phi}|_{\phi=\phi_c} = 0.$$

Expand the action

$$\int d^4x\mathcal{L}(\phi_c + \eta) = \int d^4x\mathcal{L}(\phi_c) + \frac{1}{2}\int \eta(x)\frac{\delta^2\mathcal{L}}{\delta\phi(x)\delta\phi(y)}\eta(y) + \dots$$

Thus

$$Z[J] \approx \mathrm{e}^{i\int \mathcal{L}(\phi_c)+i\int J\phi_c} \cdot \int D\eta\, \mathrm{e}^{\frac{i}{2}\int \eta \frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\eta}$$

$$Z[J] \approx \mathrm{e}^{i\int \mathcal{L}(\phi_c)+i\int J\phi_c} \cdot \det\left(\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\right)^{-1/2}$$

The second factor represents quantum (one-loop) corrections to the classical effective potential.

Continuing,

$$iF = i\int d^4x\mathcal{L}(\phi_c) + i\int J\phi_c - \frac{1}{2}\log\det\left(\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\right)^{-1/2}$$

and Γ is the same expression without the source term.

Peskin and Schroeder carry out this computation for the broken-symmetry $\mathcal{O}(N) - \sigma$ model. The result is an effective potential (computed in the $\overline{\text{MS}}$ scheme):

$$\begin{aligned} V_{eff} &= -\frac{1}{2}\mu^2\phi_c^2 + \frac{\lambda}{4}\phi_c^4 \\ &+\frac{1}{64\pi^2}\left((N-1)(\lambda\phi_c^2-\mu)^2\left[\log((\lambda\phi_c^2-\mu^2)/M^2)-\frac{3}{2}\right] + (3\lambda\phi_c^2-\mu^2)\left[\log((3\lambda\phi_c^2-\mu^2)/M^2)-\frac{3}{2}\right]\right). \end{aligned}$$

The net result is the classical Mexican hat potential with quantum corrections causing a slight distortion.

There are some subtleties to consider here

(i) the effective potential depends on μ and λ as before, but the quantum correction introduces dependence on the renormalisation scale M. This M-dependence is absorbed into the definitions of the couplings, specifically if you choose a different scale, the couplings would change in such a way as to leave the potential invariant. This is the renormalisation group we discussed last term.

(ii) as $\mu \to 0$ the effective potential simplifies to

$$V_{eff} = \frac{1}{4}\phi_c^4 \left[\lambda + \frac{\lambda^2}{16\pi^2}((N+8)\log(\lambda\phi_c^2/M^2) - \frac{3}{2}) + 9\log 3 \right].$$

This has a minimum at

$$\phi_c^2 \approx \frac{M^2}{\lambda} \mathrm{e}^{-\frac{16\pi^2}{(N+8)\lambda}}.$$

This minimum is obtained by equating leading and quantum terms and thus implies that perturbation theory is invalid.

The issue is, at least partly, resolved with the aid of the renormalisation group equations (see Peskin and Schroeder, section 13.2). Peskin and Schroeder derive the RGE that applies to the effective potential:

$$\left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} - \gamma\phi_c\frac{\partial}{\partial\phi_c} \right] V_{eff}(\phi_c; M, \lambda) = 0.$$

Solving this equation and matching it to the perturbative result (recall that these first order partial differential equations only give constraints on the solutions) gives

$$V_{eff}^{\mathrm{RGI}} = \frac{1}{4}\phi_c^4 \left[\bar\lambda + \frac{\bar\lambda^2}{16\pi^2}((N+8)(\log\bar\lambda - \frac{3}{2}) + 9\log 3) \right],$$

called the 'renormalisation group improved' effective action. $\bar\lambda$ is the running coupling we talked about last term. Notice that the new minimum is at $\phi_c = 0$ and the previous nonzero minimum was an artefact of our perturbative derivation.

Quantum Goldstone Theorem

[we follow Amit, chapter 5].

Consider the $\mathcal{O}(2)$ linear sigma model. This lagrangian has a $\mathcal{O}(2)$ symmetry under rotations of the field:

$$\phi' = \mathcal{R}\phi$$

where $\mathcal{R}$ is a 2x2 rotation matrix. For small rotation angles, ϵ

$$\mathcal{R} = 1 + \epsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since the lagrangian is invariant under this transformation (and it is on fields only), the action, the generating functional, and the effective action are also invariant. Thus

$$\begin{aligned} Z[J] &= \int D\phi\, \mathrm{e}^{i\int \mathcal{L}[\phi] + i\int J\phi} \\ &= \int D\phi'\, \mathrm{e}^{i\int \mathcal{L}[\phi'] + i\int J\phi'} \\ &= \int D\phi\, \mathrm{e}^{i\int \mathcal{L}[\phi] + i\int J\mathcal{R}\phi} \\ &\approx Z[J] + \epsilon \int D\phi \int (J_\sigma\pi - J_\pi\sigma)\, \mathrm{e}^{i\int \mathcal{L}[\phi] + i\int J\phi} \end{aligned}$$

where we have let

$$\phi = \begin{pmatrix} \pi \\ \sigma \end{pmatrix}.$$

The term proportional to ϵ must vanish and we get

$$\int \left(\frac{\delta \log Z}{\delta J_\pi} J_\sigma - \frac{\delta \log Z}{\delta J_\sigma} J_\pi \right) = 0$$

The log is there because I should have been dividing by $Z[0]$ all along. Now make the Legendre transformation (recall $J_i = -\delta\Gamma/\delta\phi_c^{(i)}$):

$$\int \left(\pi_c \frac{\delta\Gamma}{\delta\sigma_c} - \sigma_c \frac{\delta\Gamma}{\delta\pi_c} \right) = 0$$

This is a Ward identity for the linear sigma model that is true because of it's $\mathcal{O}(2)$ symmetry, regardless of whether the symmetry is hidden or not.

To get the Goldstone theorem take a derivative with respect to $\pi_c(y)$ to get

$$\int d^4x \left(\pi_c(x) \frac{\delta^2\Gamma}{\delta\pi_c(y)\delta\sigma_c(x)} + \delta(x-y) \frac{\delta\Gamma}{\delta\sigma_c(x)} - \sigma_c(x) \frac{\delta^2\Gamma}{\delta\pi_c(y)\delta\pi_c(x)} \right) = 0.$$

Finally we assume a hidden symmetry:

$$\phi_c = \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

Substituting into the last expression gives us the desired result

$$\frac{\delta\Gamma}{\delta\sigma_c(y)} - \int d^4x \, v \frac{\delta^2\Gamma}{\delta\pi_c(y)\delta\pi_c(x)} = 0$$

or

$$v \int d^4x \, \frac{\delta^2\Gamma}{\delta\pi_c(y)\delta\pi_c(x)} = J_\sigma(y)$$

As the sigma source goes to zero one must have that $v \to 0$ (but this is the symmetric vacuum, which we do not consider) or

$$\lim_{p\to 0} \Gamma^{(2)}_{\pi\pi}(p) = 0.$$

Since $\Gamma^{(2)}$ is the exact inverse propagator, this implies that the pion propagator has a pole at $p^2 = 0$ which means it is massless.