QFT — Dirac Basics: assignment 5

1. **two-component Klein-Gordon fields**

(i) A second order differential equation can be reduced to coupled first order differential equations. Do this for the free complex KG field by setting

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}$$

with $\phi_\pm = \frac{1}{\sqrt{2m}}(\pm i\partial_t + m)\psi$. If

$$i\frac{\partial \phi}{\partial t} = H\phi$$

determine the 2x2 matrix H.

(ii) Set

$$\phi^{(\pm)} = \begin{pmatrix} \chi \\ \eta \end{pmatrix} \mathrm{e}^{i(\vec{k}\cdot\vec{x}\mp Et}$$

and determine $\phi_k^{(+)}(x)$ and $\phi_{-k}^{(-)}(x)$ (up to a normalisation).

(iii) Consider charge conjugation in this approach. Let $\phi \to \phi^C = \mathcal{C}\phi^*$ where $\mathcal{C}$ is a 2x2 matrix to be determined. Find $\mathcal{C}$ such that ϕ^C obeys the same equation as ϕ and show that $\phi_{-k}^{(-)C} = \phi_k^{(+)}$. Interpret this equation.

2. **completeness**

(i) Show that

$$\sum_{ps} \left[\psi_{ps}^{(+)}(x)\psi_{ps}^{(+)\dagger}(y) + \psi_{ps}^{(-)}(x)\psi_{ps}^{(-)\dagger}(y)\right] = \delta(x-y)$$

.

(ii) Show

$$\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m.$$

3. **Diracology**

Using $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ prove

(i) $\{\gamma_5, \gamma_\mu\} = 0$

(ii) $\not{p}\,\not{p} = p^2$

(iii) $Tr(\not{a}\,\not{b}) = 4a\cdot b$.

(iv) $Tr(\text{odd no } \gamma' s) = 0$. (Hint, insert $\gamma_5^2 = 1$ in the trace.)

(v) $Tr(\not a \not b \not c \not d) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$ (Hint: use $\not a \not b + \not b \not a = 2a \cdot b$ repeatedly.)

(vi) $Tr(\gamma_5) = 0$.

(vii) $Tr(\gamma_5 \not a \not b) = 0$

(viii) $Tr(\gamma_5 \not a \not b \not c \not d) = 4i\epsilon_{\mu\nu\lambda\sigma} a^\mu b^\nu c^\lambda d^\sigma$.

(ix) $\gamma_\mu \gamma^\mu = 4$

(x) $\gamma_\mu \not a \gamma^\mu = -2 \not a$

(xi) $\gamma_\mu \not a \not b \gamma^\mu = 4a \cdot b$

(xii) $\gamma_\mu \not a \not b \not c \gamma^\mu = -2 \not c \not b \not a$

4. **conservation of total spin**

Show that $[H, \vec{J}] = 0$ where $H = \alpha \cdot \vec{p} + \beta m$ and $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$ with $\vec{\Sigma} = diag(\vec{\sigma}, \vec{\sigma})$.

5. **Lorentz invariance**

(i) Show $S^{-1}\gamma^\mu S = \Lambda^{\mu}_{\cdot\nu}\gamma^\nu$ to first order. Thus γ^μ transforms as a 4-vector.

(ii) Prove $S^{-1} = \gamma^0 S^\dagger \gamma^0$.

(iii) Prove $\gamma^5 S = S\gamma^5$.