

## QFT – assignment 1: EM Fields and Ladder Operators

1. Show that the radiation field is transverse,  $\nabla \cdot \vec{A} = 0$  and obeys the wave equation  $\nabla^2 \vec{A} - \frac{1}{c^2} \partial_t^2 \vec{A} = 0$ . Your starting point should be the mode expansion of the quantum field.

2. Prove that  $H = \sum_{k\alpha} \hbar\omega N_{k\alpha}$  and  $\vec{P} = \sum_{k\alpha} \hbar\vec{k} N_{k\alpha}$ . Use the mode expansion of the quantum field.

3. Derive an expression for the decay rate for  $A \rightarrow B\gamma$ . You may use Fermi's Golden Rule; denote the decay amplitude as  $A$  and do not evaluate it.

### 4. gauge invariance of a nonrelativistic particle coupled to photons

Prove that if  $H\psi = i\hbar\dot{\psi}$  where  $H = \frac{1}{2m}(\vec{p} - e/c\vec{A})^2 + e\phi$  then a solution to the gauge transformed Schrödinger equation (obtained by setting  $\vec{A} \rightarrow \vec{A} + \nabla\Lambda$ ,  $\phi \rightarrow \phi - \dot{\Lambda}$ ) is given by

$$\psi_\Lambda = e^{\frac{ie}{\hbar c}\Lambda(x)}\psi \quad (1)$$

Ie,  $H_\Lambda\psi_\Lambda = i\hbar\dot{\psi}_\Lambda$ .

Thus the spectrum is preserved under a gauge transformation, and the imposition of a gauge transformation requires the simultaneous rotation of the phase of the wavefunction.

Note that this result implies that the general solution to Schrödinger's equation in the presence of a vector potential can be written as

$$\psi(x) = \psi^{(0)}(x) \exp\left[\frac{ie}{\hbar c} \int^{\vec{x}} \vec{A}(\vec{y}) \cdot d\vec{s}(\vec{y})\right] \quad (2)$$

where  $\psi^{(0)}$  satisfies the Schrödinger equation with  $\vec{A} = 0$ .

### 5. coherent states [Mandl and Shaw, 1.1]

A coherent state is a minimum uncertainty wavepacket, and thus represents the 'most classical' wavefunction that can be formed. Let

$$|c\rangle = e^{-|c|^2/2} \sum_n \frac{c^n}{\sqrt{n!}} |n\rangle \quad (3)$$

where  $c$  is a complex number and  $|n\rangle = \frac{(a_{k\alpha}^\dagger)^n}{\sqrt{n!}} |0\rangle$ . Prove that

- (i)  $|c\rangle$  is normalised
- (ii)  $a_{k\alpha}|c\rangle = c|c\rangle$
- (iii)  $n = \langle c|\hat{N}|c\rangle = |c|^2$

(iv)  $\langle c|\vec{E}(\vec{x}, t)|c\rangle = -\sum_k \sqrt{\frac{2\hbar\omega}{V}} \vec{\epsilon}(k, \alpha) |c| \sin(\vec{k} \cdot \vec{x} - \omega t + \arg(c))$ . Use linear polarisation vectors.